CHAPTER 3
SIGNED NUMBERS

The positive numbers with which we have worked in previous chapters are not sufficient for every situation which may arise. For example, a negative number results in the operation of subtraction when the subtrahend is larger than the minuend.

NEGATIVE NUMBERS

When the subtrahend happens to be larger than the minuend, this fact is indicated by placing a minus sign in front of the difference, as in the following:

$$12 - 20 = -8$$

The difference, -8, is said to be NEGATIVE. A number preceded by a minus sign is a NEGATIVE NUMBER. The number -8 is read "minus eight." Such a number might arise when we speak of temperature changes. If the temperature was 12 degrees yesterday and dropped 20 degrees today, the reading today would be 12 - 20, or -8 degrees.

Numbers that show either a plus or minus sign are called SIGNED NUMBERS. An unsigned number is understood to be positive and is treated as though there were a plus sign preceding it.

If it is desired to emphasize the fact that a number is positive, a plus sign is placed in front of the number, as in +5, which is read "plus five." Therefore, either +5 or 5 indicates that the number 5 is positive. If a number is negative, a minus sign must appear in front of it, as in -9.

In dealing with signed numbers it should be emphasized that the plus and minus signs have two separate and distinct functions. They may indicate whether a number is positive or negative, or they may indicate the operation of addition or subtraction.

When operating entirely with positive numbers, it is not necessary to be concerned with this distinction since plus or minus signs indicate only addition or subtraction. However, when negative numbers are also involved in a computation, it is important to distinguish between a sign of operation and the sign of a number.

DIRECTION OF MEASUREMENT

Signed numbers provide a convenient way of indicating opposite directions with a minimum of words. For example, an altitude of 20 ft above sea level could be designated as +20 ft. The same distance below sea level would then be designated as -20 ft. One of the most common devices utilizing signed numbers to indicate direction of measurement is the thermometer.

Thermometer

The Celsius (centigrade) thermometer shown in figure 3-1 illustrates the use of positive and negative numbers to indicate direction of travel above and below 0. The 0 mark is the changeover point, at which the signs of the scale numbers change from - to +.

When the thermometer is heated by the surrounding air or by a hot liquid in which it is placed, the mercury expands and travels up the tube. After the expanding mercury passes 0, the mark at which it comes to rest is read as a positive temperature. If the thermometer is allowed to cool, the mercury contracts. After passing 0 in its downward movement, any mark at which it comes to rest is read as a negative temperature.

Rectangular Coordinate System

As a matter of convenience, mathematicians have agreed to follow certain conventions as to the use of signed numbers in directional measurement. For example, in figure 3-2, a direction to the right along the horizontal line is positive, while the opposite direction (toward the left) is negative. On the vertical line, direction upward is positive, while direction downward is negative. A distance of -3 units along the horizontal line indicates a measurement of 3 units to the left of starting point 0. A distance of -3 units on the vertical line indicates

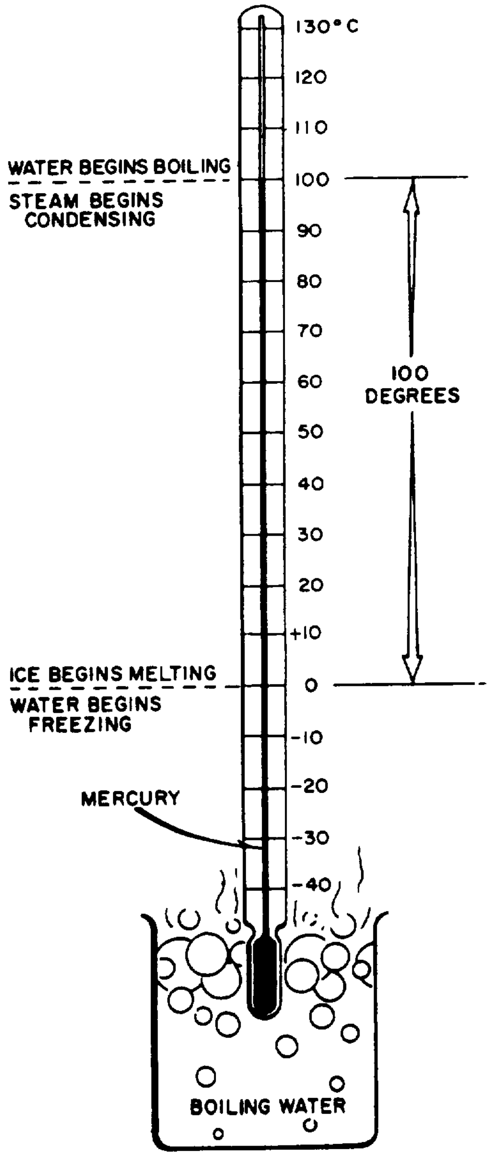

Figure 3-1.—Celsius (centigrade) temperature scale.

a measurement of 3 units below the starting point.

The two lines of the rectangular coordinate system which pass through the 0 position are the vertical axis and horizontal axis. Other vertical and horizontal lines may be included, forming a grid. When such a grid is used for the location of points and lines, the resulting "picture" containing points and lines is called a GRAPH.

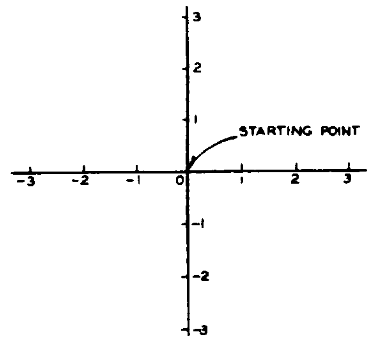

Figure 3-2.—Rectangular coordinate system.

The Number Line

Sometimes it is important to know the relative greatness (magnitude) of positive and negative numbers. To determine whether a particular number is greater or less than another number, think of all the numbers both positive and negative as being arranged along a horizontal line. (See fig. 3-3.)

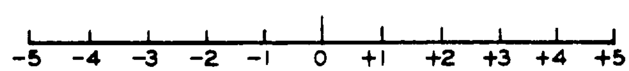

Figure 3-3.—Number line showing both positive and negative numbers.

Place zero at the middle of the line. Let the positive numbers extend from zero toward the right. Let the negative numbers extend from zero toward the left. With this arrangement, positive and negative numbers are so located that they progress from smaller to larger numbers as we move from left to right along the line. Any number that lies to the right of a given number is greater than the given number. A number that lies to the left of a given number is less than the given number. This arrangement shows that any negative number is smaller than any positive number.

The symbol for "greater than" is $>$. The symbol for "less than" is $<$. It is easy to distinguish between these symbols because the symbol used always opens toward the larger number. For example, "7 is greater than 4" can be written $7 > 4$ and "-5 is less than -1" can be written $-5 < -1$.

Absolute Value

The ABSOLUTE VALUE of a number is its numerical value when the sign is dropped. The absolute value of either +5 or -5 is 5. Thus, two numbers that differ only in sign have the same absolute value.

The symbol for absolute value consists of two vertical bars placed one on each side of the number, as in $|-5| = 5$. Consider also the following:

$$|4 - 20| = 16$$
$$|+7| = |-7| = 7$$

The expression $|-7|$ is read "absolute value of minus seven."

When positive and negative numbers are used to indicate direction of measurement, we are concerned only with absolute value, if we wish to know only the distance covered. For example, in figure 3-2, if an object moves to the left from the starting point to the point indicated by -2, the actual distance covered is 2 units. We are concerned only with the fact that $|-2| = 2$, if our only interest is in the distance and not the direction.

OPERATING WITH SIGNED NUMBERS

The number line can be used to demonstrate addition of signed numbers. Two cases must be considered; namely, adding numbers with like signs and adding numbers with unlike signs.

ADDING WITH LIKE SIGNS

As an example of addition with like signs, suppose that we use the number line (fig. 3-4) to add 2 + 3. Since these are signed numbers, we indicate this addition as (+2) + (+3). This emphasizes that, among the three + signs shown, two are number signs and one is a sign of operation. Line a (fig. 3-4) above the number line shows this addition. Find 2 on the number line. To add 3 to it, go three units more in a positive direction and get 5.

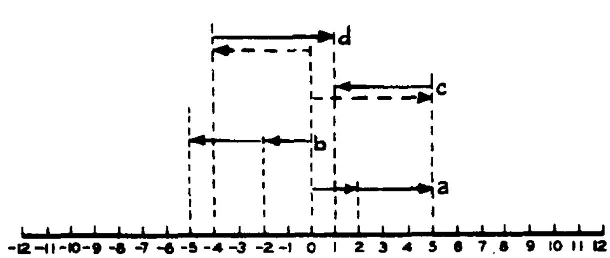

Figure 3-4.—Using the number line to add.

To add two negative numbers on the number line, such as -2 and -3, find -2 on the number line and then go three units more in the negative direction to get -5, as in b (fig. 3-4) above the number line.

Observation of the results of the foregoing operations on the number line leads us to the following conclusion, which may be stated as a law: To add numbers with like signs, add the absolute values and prefix the common sign.

ADDING WITH UNLIKE SIGNS

To add a positive and a negative number, such as (-4) + (+5), find +5 on the number line and go four units in a negative direction, as in line c above the number line in figure 3-4. Notice that this addition could be performed in the other direction. That is, we could start at -4 and move 5 units in the positive direction. (See line d, fig. 3-4.)

The results of our operations with mixed signs on the number line lead to the following conclusion, which may be stated as a law: To add numbers with unlike signs, find the difference between their absolute values and prefix the sign of the numerically greater number.

The following examples show the addition of the numbers 3 and 5 with the four possible combinations of signs:

3	-3	3	-3
5	-5	-5	5
8	-8	-2	2

In the first example, 3 and 5 have like signs and the common sign is understood to be positive. The sum of the absolute values is 8 and no sign is prefixed to this sum, thus signifying that the sign of the 8 is understood to be positive.

In the second example, the 3 and 5 again have like signs, but their common sign is negative. The sum of the absolute values is 8, and this time the common sign is prefixed to the sum. The answer is thus -8.

In the third example, the 3 and 5 have unlike signs. The difference between their absolute values is 2, and the sign of the larger addend is negative. Therefore, the answer is -2.

In the fourth example, the 3 and 5 again have unlike signs. The difference of the absolute values is still 2, but this time the sign of the larger addend is positive. Therefore, the sign prefixed to the 2 is positive (understood) and the final answer is simply 2.

These four examples could be written in a different form, emphasizing the distinction between the sign of a number and an operational sign, as follows:

$$(+3) + (+5) = +8$$
$$(-3) + (-5) = -8$$
$$(+3) + (-5) = -2$$
$$(-3) + (+5) = +2$$

Practice problems. Add as indicated:

1. $-10 + 5 = (-10) + (+5) = ?$
2. Add -9, -16, and 25
3. $-7 - 1 - 3 = (-7) + (-1) + (-3) = ?$
4. Add -22 and -13

Answers:

1. -5
2. 0
3. -11
4. -35

SUBTRACTION

Subtraction is the inverse of addition. When subtraction is performed, we "take away" the subtrahend. This means that whatever the value of the subtrahend, its effect is to be reversed when subtraction is indicated. In addition, the sum of 5 and -2 is 3. In subtraction, however, to take away the effect of the -2, the quantity +2 must be added. Thus the difference between +5 and -2 is +7.

Keeping this idea in mind, we may now proceed to examine the various combinations of subtraction involving signed numbers. Let us first consider the four possibilities where the minuend is numerically greater than the subtrahend, as in the following examples:

8	8	-8	-8
5	-5	5	-5
3	13	-13	-3

We may show how each of these results is obtained by use of the number line, as shown in figure 3-5.

In the first example, we find +8 on the number line, then subtract 5 by making a movement that reverses its sign. Thus, we move to the left 5 units. The result (difference) is +3. (See line a, fig. 3-5.)

In the second example, we find +8 on the number line, then subtract (-5) by making a movement that will reverse its sign. Thus we move to the right 5 units. The result in this case is +13. (See line b, fig. 3-5.)

In the third example, we find -8 on the number line, then subtract 5 by making a movement that reverses its sign. Thus we move to the left 5 units. The result is -13. (See line c, fig. 3-5.)

In the fourth example, we find -8 on the number line, then reverse the sign of -5 by moving 5 units to the right. The result is -3. (See line d, fig. 3-5.)

Next, let us consider the four possibilities that arise when the subtrahend is numerically greater than the minuend, as in the following examples:

5	5	-5	-5
8	-8	8	-8
-3	13	-13	3

In the first example, we find +5 on the number line, then subtract 8 by making a movement

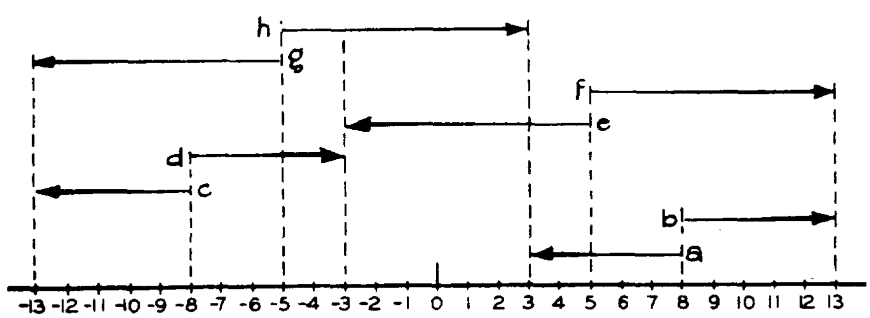

Figure 3-5.—Subtraction by use of the number line.

that reverses its sign. Thus we move to the left 8 units. The result is -3. (See line e, fig. 3-5.)

In the second example, we find +5 on the number line, then subtract -8 by making a movement to the right that reverses its sign. The result is 13. (See line f, fig. 3-5.)

In the third example, we find -5 on the number line, then reverse the sign of 8 by a movement to the left. The result is -13. (See line g, fig. 3-5.)

In the fourth example, we find -5 on the number line, then reverse the sign of -8 by a movement to the right. The result is 3. (See line h, fig. 3-5.)

Careful study of the preceding examples leads to the following conclusion, which is stated as a law for subtraction of signed numbers: In any subtraction problem, mentally change the sign of the subtrahend and proceed as in addition.

Practice problems. In problems 1 through 4, subtract the lower number from the upper. In 5 through 8, subtract as indicated.

1. 17	2. -12	3. -9	4. 7
-10	8	-13	16

5. 1 -(-5) = ?
6. -6 -(-8) = ?
7. 14 - 7 -(-3) = ?
8. -9 - 2 = ?

Answers:

1. 27	2. -20	3. 4	4. -9
5. 6	6. 2	7. 10	8. -11

MULTIPLICATION

To explain the rules for multiplication of signed numbers, we recall that multiplication of whole numbers may be thought of as shortened addition. Two types of multiplication problems must be examined; the first type involves numbers with unlike signs, and the second involves numbers with like signs.

Unlike Signs

Consider the example 3(-4), in which the multiplicand is negative. This means we are to add -4 three times; that is, 3(-4) is equal to (-4) + (-4) + (-4), which is equal to -12. For example, if we have three 4-dollar debts, we owe 12 dollars in all.

When the multiplier is negative, as in -3(7), we are to take away 7 three times. Thus, -3(7) is equal to -(7) - (7) - (7) which is equal to -21. For example, if 7 shells were expended in one firing, 7 the next, and 7 the next, there would be a loss of 21 shells in all. Thus, the rule is as follows: The product of two numbers with unlike signs is negative.

The law of signs for unlike signs is sometimes stated as follows: Minus times plus is minus; plus times minus is minus. Thus a problem such as 3(-4) can be reduced to the following two steps:

1. Multiply the signs and write down the sign of the answer before working with the numbers themselves.
2. Multiply the numbers as if they were unsigned numbers.

Using the suggested procedure, the sign of the answer for 3(-4) is found to be minus. The product of 3 and 4 is 12, and the final answer is -12. When there are more than two numbers to be multiplied, the signs are taken in pairs until the final sign is determined.

Like Signs

When both factors are positive, as in 4(5), the sign of the product is positive. We are to add +5 four times, as follows:

$$4(5) = 5 + 5 + 5 + 5 = 20$$

When both factors are negative, as in -4(-5), the sign of the product is positive. We are to take away -5 four times.

$$\begin{aligned} -4(-5) &= -(-5) - (-5) - (-5) - (-5) \\ &= +5 +5 +5 +5 \\ &= 20 \end{aligned}$$

Remember that taking away a negative 5 is the same as adding a positive 5. For example, suppose someone owes a man 20 dollars and pays him back (or diminishes the debt) 5 dollars at a time. He takes away a debt of 20 dollars by giving him four positive 5-dollar bills, or a total of 20 positive dollars in all.

The rule developed by the foregoing example is as follows: The product of two numbers with like signs is positive.

Knowing that the product of two positive numbers or two negative numbers is positive, we can conclude that the product of any even number of negative numbers is positive. Similarly, the product of any odd number of negative numbers is negative.

The laws of signs may be combined as follows: Minus times plus is minus; plus times minus is minus; minus times minus is plus; plus times plus is plus. Use of this combined rule may be illustrated as follows:

$$4(-2) \cdot (-5) \cdot (6) \cdot (-3) = -720$$

Taking the signs in pairs, the understood plus on the 4 times the minus on the 2 produces a minus. This minus times the minus on the 5 produces a plus. This plus times the understood plus on the 6 produces a plus. This plus times the minus on the 3 produces a minus, so we know that the final answer is negative. The product of the numbers, disregarding their signs, is 720; therefore, the final answer is -720.

Practice problems. Multiply as indicated:

1. $5(-8) = ?$
2. $-7(3)(2) = ?$
3. $6(-1)(-4) = ?$
4. $-2(3)(-4)(5)(-6) = ?$

Answers:

1. -40	3. 24
2. -42	4. -720

DIVISION

Because division is the inverse of multiplication, we can quickly develop the rules for division of signed numbers by comparison with the corresponding multiplication rules, as in the following examples:

1. Division involving two numbers with unlike signs is related to multiplication with unlike signs, as follows:

$$3(-4) = -12$$

Therefore, $$\frac{-12}{3} = -4$$

Thus, the rule for division with unlike signs is: The quotient of two numbers with unlike signs is negative.

2. Division involving two numbers with like signs is related to multiplication with like signs, as follows:

$$3(-4) = -12$$

Therefore, $$\frac{-12}{-4} = 3$$

Thus the rule for division with like signs is: The quotient of two numbers with like signs is positive.

The following examples show the application of the rules for dividing signed numbers:

$$\frac{12}{3} = 4 \qquad \frac{-12}{3} = -4$$

$$\frac{-12}{-3} = 4 \qquad \frac{12}{-3} = -4$$

Practice problems. Multiply and divide as indicated:

1. $15 \div -5$	3. $\frac{(-3)(4)}{-6}$
2. $-2(-3)/-6$	4. $-81/9$

Answers:

1. -3	3. 2
2. -1	4. -9

SPECIAL CASES

Two special cases arise frequently in which the laws of signs may be used to advantage. The first such usage is in simplifying subtraction; the second is in changing the signs of the numerator and denominator when division is indicated in the form of a fraction.

Subtraction

The rules for subtraction may be simplified by use of the laws of signs, if each expression to be subtracted is considered as being multiplied by a negative sign. For example, $4 -(-5)$ is the same as $4 + 5$, since minus times minus is plus. This result also establishes a basis for the rule governing removal of parentheses.

The parentheses rule, as usually stated, is: Parentheses preceded by a minus sign may be removed, if the signs of all terms within the parentheses are changed. This is illustrated as follows:

$$12 -(3 - 2 + 4) = 12 - 3 + 2 - 4$$

The reason for the changes of sign is clear when the negative sign preceding the parentheses is considered to be a multiplier for the whole parenthetical expression.

Division in Fractional Form

Division is often indicated by writing the dividend as the numerator, and the divisor as the denominator, of a fraction. In algebra, every fraction is considered to have three signs. The numerator has a sign, the denominator has a sign, and the fraction itself, taken as a whole, has a sign. In many cases, one or more of these signs will be positive, and thus will not be shown. For example, in the following fraction the sign of the numerator and the sign of the denominator are both positive (understood) and the sign of the fraction itself is negative:

$$-\frac{4}{5}$$

Fractions with more than one negative sign are always reducible to a simpler form with at most one negative sign. For example, the sign of the numerator and the sign of the denominator may be both negative. We note that minus divided by minus gives the same result as plus divided by plus. Therefore, we may change to the less complicated form having plus signs (understood) for both numerator and denominator, as follows:

$$\frac{-15}{-5} = \frac{+15}{+5} = \frac{15}{5}$$

Since -15 divided by -5 is 3, and 15 divided by 5 is also 3, we conclude that the change of sign does not alter the final answer. The same reasoning may be applied in the following example, in which the sign of the fraction itself is negative:

$$-\frac{-15}{-5} = -\frac{+15}{+5} = -\frac{15}{5}$$

When the fraction itself has a negative sign, as in this example, the fraction may be enclosed in parentheses temporarily, for the purpose of working with the numerator and denominator only. Then the sign of the fraction is applied separately to the result, as follows:

$$-\frac{-15}{-5} = -\left(\frac{-15}{-5}\right) = -(3) = -3$$

All of this can be done mentally.

If a fraction has a negative sign in one of the three sign positions, this sign may be moved to another position. Such an adjustment is an advantage in some types of complicated expressions involving fractions. Examples of this type of sign change follow:

$$-\frac{15}{5} = \frac{-15}{5} = \frac{15}{-5}$$

In the first expression of the foregoing example, the sign of the numerator is positive (understood) and the sign of the fraction is negative. Changing both of these signs, we obtain the second expression. To obtain the third expression from the second, we change the sign of the numerator and the sign of the denominator. Observe that the sign changes in each case involve a pair of signs. This leads to the law of signs for fractions: Any two of the three signs of a fraction may be changed without altering the value of the fraction.

AXIOMS AND LAWS

An axiom is a self-evident truth. It is a truth that is so universally accepted that it does not require proof. For example, the statement that "a straight line is the shortest distance between two points" is an axiom from plane geometry. One tends to accept the truth of an axiom without proof, because anything which is axiomatic is, by its very nature, obviously true. On the other hand, a law (in the mathematical sense) is the result of defining certain quantities and relationships and then developing logical conclusions from the definitions.

AXIOMS OF EQUALITY

The four axioms of equality with which we are concerned in arithmetic and algebra are stated as follows:

1. If the same quantity is added to each of two equal quantities, the resulting quantities are equal. This is sometimes stated as follows: If equals are added to equals, the results are equal. For example, by adding the same quantity (3) to both sides of the following equation, we obtain two sums which are equal:

$$\begin{aligned} -2 &= -3 + 1 \\ -2 + 3 &= -3 + 1 + 3 \\ 1 &= 1 \end{aligned}$$

2. If the same quantity is subtracted from each of two equal quantities, the resulting quantities are equal. This is sometimes stated as follows: If equals are subtracted from equals, the results are equal. For example, by subtracting 2 from both sides of the following equation we obtain results which are equal:

$$\begin{aligned} 5 &= 2 + 3 \\ 5 - 2 &= 2 + 3 - 2 \\ 3 &= 3 \end{aligned}$$

3. If two equal quantities are multiplied by the same quantity, the resulting products are equal. This is sometimes stated as follows: If equals are multiplied by equals, the products are equal. For example, both sides of the following equation are multiplied by -3 and equal results are obtained:

$$\begin{aligned} 5 &= 2 + 3 \\ (-3)(5) &= (-3)(2 + 3) \\ -15 &= -15 \end{aligned}$$

4. If two equal quantities are divided by the same quantity, the resulting quotients are equal. This is sometimes stated as follows: If equals are divided by equals, the results are equal. For example, both sides of the following equation are divided by 3, and the resulting quotients are equal:

$$12 + 3 = 15$$

$$\frac{12 + 3}{3} = \frac{15}{3}$$

$$4 + 1 = 5$$

These axioms are especially useful when letters are used to represent numbers. If we know that $5x = -30$, for instance, then dividing both $5x$ and -30 by 5 leads to the conclusion that $x = -6$.

LAWS FOR COMBINING NUMBERS

Numbers are combined in accordance with the following basic laws:

1. The associative laws of addition and multiplication.
2. The commutative laws of addition and multiplication.
3. The distributive law.

Associative Law of Addition

The word "associative" suggests association or grouping. This law states that the sum of three or more addends is the same regardless of the manner in which they are grouped. For example, $6 + 3 + 1$ is the same as $6 + (3 + 1)$ or $(6 + 3) + 1$.

This law can be applied to subtraction by changing signs in such a way that all negative signs are treated as number signs rather than operational signs. That is, some of the addends can be negative numbers. For example, $6 - 4 - 2$ can be rewritten as $6 + (-4) + (-2)$. By the associative law, this is the same as

$$6 + [(-4) + (-2)] \text{ or } [6 + (-4)] + (-2).$$

However, $6 - 4 - 2$ is not the same as $6 - (4 - 2)$; the terms must be expressed as addends before applying the associative law of addition.

Associative Law of Multiplication

This law states that the product of three or more factors is the same regardless of the manner in which they are grouped. For example, $6 \cdot 3 \cdot 2$ is the same as $(6 \cdot 3) \cdot 2$ or $6 \cdot (3 \cdot 2)$. Negative signs require no special treatment in the application of this law. For example, $6 \cdot (-4) \cdot (-2)$ is the same as $[6 \cdot (-4)] \cdot (-2)$ or $6 \cdot [(-4) \cdot (-2)]$.

Commutative Law of Addition

The word "commute" means to change, substitute or move from place to place. The commutative law of addition states that the sum of two or more addends is the same regardless of the order in which they are arranged. For example, $4 + 3 + 2$ is the same as $4 + 2 + 3$ or $2 + 4 + 3$.

This law can be applied to subtraction by changing signs so that all negative signs become number signs and all signs of operation are positive. For example, $5 - 3 - 2$ is changed to $5 + (-3) + (-2)$, which is the same as $5 + (-2) + (-3)$ or $(-3) + 5 + (-2)$.

Commutative Law of Multiplication

This law states that the product of two or more factors is the same regardless of the order in which the factors are arranged. For example, $3 \cdot 4 \cdot 5$ is the same as $5 \cdot 3 \cdot 4$ or

$4 \cdot 3 \cdot 5$. Negative signs require no special treatment in the application of this law. For example, $2 \cdot (-4) \cdot (-3)$ is the same as $(-4) \cdot (-3) \cdot 2$ or $(-3) \cdot 2 \cdot (-4)$.

Distributive Law

This law combines the operations of addition and multiplication. The word "distributive" refers to the distribution of a common multiplier among the terms of an additive expression. For example,

$$\begin{aligned} 2(3 + 4 + 5) &= 2 \cdot 3 + 2 \cdot 4 + 2 \cdot 5 \\ &= 6 + 8 + 10 \end{aligned}$$

To verify the distributive law, we note that $2(3 + 4 + 5)$ is the same as $2(12)$ or 24. Also, $6 + 8 + 10$ is 24. For application of the distributive law where negative signs appear, the following procedure is recommended:

$$\begin{aligned} 3(4 - 2) &= 3\,[4 + (-2)] \\ &= 3(4) + 3(-2) \\ &= 12 - 6 \\ &= 6 \end{aligned}$$